The aſhes of this foſſil, when burnt, being boiled in water, and the water evaporated, there remained no ſalt behind.

I am, my Lord, *&c.*

Groſvenor-Street, Feb. 28, 1760.

LIV. *A new Method of computing the Sums of certain Series; by Mr.* John Landen: *Communicated by Mr.* Thomas Simpſon, *F. R. S.*

Read Feb. 28, 1760.

AS the improving the analytic art, eſpecially any branch of it that relates to the ſummation of ſeries, may, by facilitating computations, conduce to the improvement of ſeveral branches of ſcience; it is preſumed, that this paper, which exhibits a new and eaſy method of computing the ſums of a great number of infinite ſeries, may be acceptable to the mathematical world, and deemed worthy to be inſerted in the Britiſh Philoſophical Tranſactions.

1.

Suppoſing x to be the ſine of the circular arc z, whoſe radius is 1, $\frac{\dot{x}}{\sqrt{1-x^2}}$ will be $=\dot{z}$; and, conſequently, $\frac{\dot{x}}{\sqrt{x^2-1}} = \frac{\dot{z}}{\sqrt{-1}}$. From whence, by taking the correct fluents, we have hyp. log. $\frac{x+\sqrt{x^2-1}}{\sqrt{-1}} = \frac{z}{\sqrt{-1}}$.

Hence,

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Hence, writing a for one fourth of the periphery of the circle whoſe radius is 1, and taking x equal to the ſaid radius, we find hyp. log. $\frac{1}{\sqrt{-1}} = \frac{a}{\sqrt{-1}}$; and, conſequently, hyp. log. $\sqrt{-1} = \frac{-a}{\sqrt{-1}}$, and hyp. log. $-1 = \pm \frac{2a}{\sqrt{-1}}$.

2.

The hyp. log. of $\frac{1}{1-x}$ being $= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$, &c.

$\overset{\text{I}}{\text{F}} =$ fluent of $\frac{\dot{x}}{x}$ hyp. log. $\frac{1}{1-x}$, is $= x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$ $+ \frac{x^4}{4^2}$, &c.

$\overset{\text{II}}{\text{F}} =$ fluent of $\frac{\dot{x}}{x} \overset{\text{I}}{\text{F}} = x + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \frac{x^4}{4^3}$, &c.

$\overset{\text{III}}{\text{F}} =$ fluent of $\frac{\dot{x}}{x} \overset{\text{II}}{\text{F}} = x + \frac{x^2}{2^4} + \frac{x^3}{3^4} + \frac{x^4}{4^4}$, &c.

$\overset{\text{IV}}{\text{F}} =$ fluent of $\frac{\dot{x}}{x} \overset{\text{III}}{\text{F}} = x + \frac{x^2}{2^5} + \frac{x^3}{3^5} + \frac{x^4}{4^5}$, &c.

&c. &c. &c.

3.

By writing, in the firſt equation in the preceding article, $\frac{1}{x}$ inſtead of x, we have

Hyp. log. $\frac{1}{1-\frac{1}{x}} = x^{-1} + \frac{x^{-2}}{2} + \frac{x^{-3}}{3}$, &c.

But the hyp. log. of $\frac{1}{1-\frac{1}{x}}$ is $=$ hyp. log. $\frac{x}{x-1} =$

hyp.

hyp. log. $\frac{1}{1-x}$ + hyp. log. x + hyp. log. -1 = $\pm 2b + X +$ hyp. log. $\frac{1}{1-x}$, b being put for $\frac{a}{\sqrt{-1}}$, and X for the hyp. log. of x.

It is evident, therefore, that

Hyp. log. $\frac{1}{1-x}$ is $= \mp 2b - X + x^{-1} + \frac{x^{-2}}{2} + \frac{x^{-3}}{3}$, &c. where, of the two ſigns prefixed to $2b$, the upper one takes place, when the hyp. log. of -1 is taken equal to $\frac{-2a}{\sqrt{-1}}$, likewiſe when x is taken equal to $\sqrt{-1}$; and the lower one takes place, when the hyp. log. of -1 is taken equal to $\frac{2a}{\sqrt{-1}}$, alſo when x is taken equal to $\frac{1}{\sqrt{-1}}$: wherefore, if we obſerve to take the value of hyp. log. of -1, as laſt mentioned, and x equal to $\frac{1}{\sqrt{-1}}$, inſtead of $\sqrt{-1}$, we need retain only the lower of the ſaid ſigns.

4.

For brevity ſake, we ſhall, in what follows, put the ſeries $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$, &c. $= \overset{II}{P}$,

$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4}$, &c. $= \overset{IV}{P}$,

$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6}$, &c. $= \overset{VI}{P}$,

&c. &c.

$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}$, &c. $= \overset{II}{Q}$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} +, \text{\&c.} = \overset{\text{III}}{q},$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4}, \text{\&c.} = \overset{\text{IV}}{Q}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} +, \text{\&c.} = \overset{\text{V}}{q},$$

&c. &c.

5.

Multiplying the laſt equation in art. 3. by $\frac{\dot{x}}{x}$, and taking the correct fluents, we have

$$\overset{\text{I}}{F} = 2\overset{\text{II}}{P} + 2bX - \frac{X^2}{2} - x^{-1} - \frac{x^{-2}}{2^2} - \frac{x^{-3}}{3^2}, \text{\&c.}$$

From whence, by multiplying by $\frac{\dot{x}}{x}$, and taking the fluents, we get

$$\overset{\text{II}}{F} = 2\overset{\text{II}}{P}X + bX^2 - \frac{X^3}{2.3} + x^{-1} + \frac{x^{-2}}{2^3} + \frac{x^{-3}}{3^3}, \text{\&c.}$$

Again, multiplying the laſt equation by $\frac{\dot{x}}{x}$, and taking the correct fluents, we find

$$\overset{\text{III}}{F} = 2\overset{\text{IV}}{P} + \overset{\text{II}}{P}X^2 + \frac{bX^3}{3} - \frac{X^4}{2.3.4} - x^{-1} - \frac{x^{-2}}{2^4} - \frac{x^{-3}}{3^4}, \text{\&c.}$$

And, by proceeding in the ſame manner, we find

$$\overset{\text{IV}}{F} = 2\overset{\text{IV}}{P}X + \frac{\overset{\text{II}}{P}X^3}{3} + \frac{bX^4}{3.4} - \frac{X^5}{2.3.4.5} + x^{-1} + \frac{x^{-2}}{2^5} + \frac{x^{-3}}{3^5}, \text{\&c.}$$

&c. &c.

6.

Now, it is obvious, that $x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. the value of $\overset{\text{I}}{F}$ in art. 2. muſt be equal to $2\overset{\text{II}}{P} + 2bX$

—

$-\frac{X^2}{2} - x^{-1} - \frac{x^{-2}}{2^2} - \frac{x^{-3}}{3^2}$, &c. the value of F in art. 5. when both ſeries converge.

Therefore, $\frac{x + x^{-1}}{1^2} + \frac{x^2 + x^{-2}}{2^2} + \frac{x^3 + x^{-3}}{3^2}$, &c. is then $= 2\overset{II}{P} + 2bX - \frac{X^2}{2}$.

From which equation, by taking x equal to -1, we have $-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} -$, &c. $= \overset{II}{P} + b^2 = \overset{II}{P} - a^2$; and, by taking x equal to $\frac{1}{\sqrt{-1}}$, we have $-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} -$, &c. $= 4\overset{II}{P} + 3b^2 = 4\overset{II}{P} - 3a^2$.

Therefore $4\overset{II}{P} - 3a^2$ is $= \overset{II}{P} - a^2$:

Hence $\overset{II}{P}$ is found $= \frac{2a^2}{3}$.

Moreover $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$, &c. being $= \overset{II}{P}$, by ſuppoſition; and $-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} -$, &c. $= \overset{II}{P} - a^2$, as found above; we, by ſubtraction, get $\frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2}$, &c. $(= 2\overset{II}{Q}) = a^2$, and, conſequently $\overset{II}{Q} = \frac{a^2}{2}$.

SCHOLIUM.

The hyp. log. of $\frac{1}{1 - x}$ being $= x + \frac{x^2}{2} + \frac{x^3}{3}$, &c. we, by writing $1 - x$ inſtead of x, have

Hyp. log. of $\frac{1}{x} = 1 - x + \frac{\overline{1-x}|^{2}}{2} + \frac{\overline{1-x}|^{3}}{3}$, &c. and conſequently $X = -1 - x - \frac{\overline{1-x}|^{2}}{2} - \frac{\overline{1-x}|^{3}}{3}$, &c.

Moreover the fluent of $\frac{\dot{x}}{x} \times$ hyp. log. of $\frac{1}{1-x}$ is $= x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. which vaniſhes when x vaniſhes; and the fluent of $\frac{\dot{x}}{1-x} \times X$ is $= \overline{1-x} + \frac{\overline{1-x}|^{2}}{2^2} + \frac{\overline{1-x}|^{3}}{3^2}$, &c. $- \overset{\text{II}}{P}$, being corrected ſo as to vaniſh when x vaniſhes.

But the fluent of $\frac{\dot{x}}{x} \times$ hyp. log. of $\frac{1}{1-x}$ + fluent of $\frac{\dot{x}}{1-x} \times X$ is $= X \times$ hyp. log. of $\frac{1}{1-x}$, which alſo vaniſhes when x vaniſhes.

Therefore $X \times$ hyp. log. of $\frac{1}{1-x}$ is $= x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. $+ 1 - x + \frac{\overline{1-x}|^{2}}{2^2} + \frac{\overline{1-x}|^{3}}{3^2}$, &c. $- \overset{\text{II}}{P}$.

From whence, by taking x equal to $\frac{1}{2}$, we find − ſquare of hyp. log. of $2 = 2 \times \frac{1}{1^2.2^1} + \frac{1}{2^2.2^2} + \frac{1}{3^2.2^3}$, &c. $- \overset{\text{II}}{P}$: hence, $\overset{\text{II}}{P}$ being before found $= \frac{2a^2}{3}$, it appears that, when x is $= \frac{1}{2}$, the ſeries $x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. is $= \frac{a^2}{3} - \frac{1}{2} \times \overline{\text{hyp. log. of } 2}|^{2}$.

7.

Furthermore, $x + \frac{x^2}{2^3} + \frac{x^3}{3^3}$, &c. the value of $\overset{\text{II}}{F}$ in art. 2. muſt be equal to $2\overset{\text{II}}{P}X + bX^2 - \frac{X^3}{2.3} + x^{-1}$

$+\frac{x^{-2}}{2^3}+\frac{x^{-3}}{3^3}$, &c. the value of $\overset{\text{II}}{F}$ in art. 5. when both ſeries converge.

Therefore $\frac{x-x^{-1}}{1^3}+\frac{x^2-x^{-2}}{2^3}+\frac{x^3-x^{-3}}{3^3}$, &c. is then $=2\overset{\text{II}}{P}X+bX^2-\frac{X^3}{2.3}$.

From whence, by taking x equal to -1, we have $4b\overset{\text{II}}{P}+4b^3-\frac{8b^3}{2.3}=0$; and, conſequently, $\overset{\text{II}}{P}=\frac{2a^2}{3}$, as before found.

And, by taking x equal to $\frac{1}{\sqrt{-1}}$, we find

$$\frac{2}{\sqrt{-1}}\times\overset{\text{III}}{q}=2b\overset{\text{II}}{P}+b^3-\frac{b^3}{2.3}=\frac{4a^3}{3\sqrt{-1}}-\frac{a^3}{\sqrt{-1}}+\frac{a^3}{2.3\sqrt{-1}}$$

$$=\frac{a^3}{2\sqrt{-1}}.$$

Therefore $\overset{\text{III}}{q}$ is $=\frac{a^3}{4}$.

8.

From what is done above, it evidently follows, that

$$-\overset{\text{IV}}{P}\text{ is }=\frac{2b^2\overset{\text{II}}{P}}{3}+\frac{2.8b^4}{3.4.5},$$

$$-\overset{\text{VI}}{P}=\frac{2b^2\overset{\text{IV}}{P}}{3}+\frac{8b^4\overset{\text{II}}{P}}{3.4.5}+\frac{3.32b^6}{3.4.5.6.7},$$

&c. &c.

$$-\overset{\text{IV}}{Q}=b^2\overset{\text{II}}{P}+\frac{3.2b^4}{3.4},$$

$$-\overset{\text{VI}}{Q}=b^2\overset{\text{IV}}{P}+\frac{4b^4\overset{\text{II}}{P}}{3.4}+\frac{5.8b^6}{3.4.5.6},$$

&c. &c.

$\frac{\overset{V}{q}}{\sqrt{-1}}$ is $= b\overset{IV}{P} + \frac{b^3\overset{II}{P}}{2.3} + \frac{9b^5}{2.2.3.4.5}$,

$\frac{\overset{VII}{q}}{\sqrt{-1}} = b\overset{VI}{P} + \frac{b^3\overset{IV}{P}}{2.3} + \frac{b^5\overset{II}{P}}{2.3.4.5} + \frac{13b^7}{2.2.3.4.5.6.7}$,

&c. &c.

From whence the values of $\overset{IV}{P}$, $\overset{VI}{P}$, &c. $\overset{IV}{Q}$, $\overset{VI}{Q}$, &c. $\overset{V}{q}$, $\overset{VII}{q}$, &c. may be eaſily obtained, in terms of a.

9.

Hyp. log. $\left.\overline{\frac{1+x}{1-x}}\right|^{\frac{1}{2}}$ being $= x + \frac{x^3}{3} + \frac{x^5}{5}$, &c.

$\overset{I}{G} =$ fluent of $\frac{\dot{x}}{x}$ hyp. log. $\left.\overline{\frac{1+x}{1-x}}\right|^{\frac{1}{2}}$ is $= x + \frac{x^3}{3^2} + \frac{x^5}{5^2}$, &c.

$\overset{II}{G} =$ fluent of $\frac{\dot{x}}{x}\overset{I}{G} = x + \frac{x^3}{3^3} + \frac{x^5}{5^3}$, &c.

$\overset{III}{G} =$ fluent of $\frac{\dot{x}}{x}\overset{II}{G} = x + \frac{x^3}{3^4} + \frac{x^5}{5^4}$, &c.

&c. &c. &c.

10.

By writing, in the firſt equation in the preceding article, $\frac{1}{x}$ inſtead of x, we have

$$\text{Hyp. log. } \left.\overline{\frac{1+\frac{1}{x}}{1-\frac{1}{x}}}\right|^{\frac{1}{2}} = x^{-1} + \frac{x^{-3}}{3} + \frac{x^{-5}}{5}, \text{ \&c.}$$

But the hyp. log. of $\left.\overline{\frac{1+\frac{1}{x}}{1-\frac{1}{x}}}\right|^{\frac{1}{2}}$ is $=$ hyp. log. $\left.\overline{\frac{x+1}{x-1}}\right|^{\frac{1}{2}}$ $=$ hyp. log. $\left.\overline{\frac{1+x}{1-x}}\right|^{\frac{1}{2}} \mp$ hyp. log. $\sqrt{-1} = \pm b +$ hyp. log. $\left.\overline{\frac{1+x}{1-x}}\right|^{\frac{1}{2}}$.

It

It is manifeſt, therefore, that

Hyp. log. $\overline{\frac{1+x}{1-x}}\Big|^{\frac{1}{2}}$ is $= \mp b + x^{-1} + \frac{x^{-3}}{3} + \frac{x^{-5}}{5}$, &c. where, with reſpect to the two ſigns prefixed to b, the ſame obſervation may be made as in art. 3.

11.

Multiplying the laſt equation by $\frac{\dot{x}}{x}$, and taking the correct fluents, we have

$$\overset{\mathrm{I}}{G} = 2\overset{\mathrm{II}}{Q} + bX - x^{-1} - \frac{x^{-3}}{3^2} - \frac{x^{-5}}{5^2}, \text{ \&c.}$$

From whence, by multiplying by $\frac{\dot{x}}{x}$, and taking the fluents, we get

$$\overset{\mathrm{II}}{G} = 2\overset{\mathrm{II}}{Q}X + \frac{bX^2}{2} + x^{-1} + \frac{x^{-3}}{3^3} + \frac{x^{-5}}{5^3}, \text{ \&c.}$$

Again, multiplying the laſt equation by $\frac{\dot{x}}{x}$, and taking the correct fluents, we find

$$\overset{\mathrm{III}}{G} = 2\overset{\mathrm{IV}}{Q} + \overset{\mathrm{II}}{Q}X^2 + \frac{bX^3}{2.3} - x^{-1} - \frac{x^{-3}}{3^4} - \frac{x^{-5}}{5^4}, \text{ \&c.}$$

And, by proceeding in the ſame manner, we find

$$\overset{\mathrm{IV}}{G} = 2\overset{\mathrm{IV}}{Q}X + \frac{\overset{\mathrm{II}}{Q}X^3}{3} + \frac{bX^4}{2.3.4} + x^{-1} + \frac{x^{-3}}{3^5} + \frac{x^{-5}}{5^5}, \text{ \&c.}$$

&c. &c.

12.

Now, it is obvious, that $x + \frac{x^3}{3^2} + \frac{x^5}{5^2}$, &c. the value of $\overset{\mathrm{I}}{G}$ in art. 9. muſt be equal to $2\overset{\mathrm{II}}{Q} + bX - x^{-1} - \frac{x^{-3}}{3^2} - \frac{x^{-5}}{5^2}$, &c. the value of $\overset{\mathrm{I}}{G}$ in art. 11. when both ſeries converge

There-

Therefore $\frac{x + x^{-1}}{1^2} + \frac{x^3 + x^{-3}}{3^2} + \frac{x^5 + x^{-5}}{5^2}$, &c. is then $= 2\overset{\text{II}}{Q} + bX$.

From whence, by taking x equal to $\frac{1}{\sqrt{-1}}$, we have $2\overset{\text{II}}{Q} + b^2 = 0$; and, conſequently, $\overset{\text{II}}{Q} = \frac{a^2}{2}$, as in art. 6.

13.

Likewiſe $x + \frac{x^3}{3^3} + \frac{x^5}{5^3}$, &c. the value of $\overset{\text{II}}{G}$ in art. 9. muſt be equal to $2\overset{\text{II}}{Q}X + \frac{bX^2}{2} + x^{-1} + \frac{x^{-3}}{3^3} + \frac{x^{-5}}{5^3}$, &c. the value of $\overset{\text{II}}{G}$ in art. 11. when both ſeries converge.

Therefore $\frac{x - x^{-1}}{1^3} + \frac{x^3 - x^{-3}}{3^3} + \frac{x^5 - x^{-5}}{5^3}$, &c. is then $= 2\overset{\text{II}}{Q}X + \frac{bX^2}{2}$.

Hence, by taking $x = \frac{1}{\sqrt{-1}}$, we find $\frac{2}{\sqrt{-1}} \times \overset{\text{III}}{q} = 2b\overset{\text{II}}{Q} + \frac{b^3}{2} = \frac{a^3}{2\sqrt{-1}}$; and, conſequently, $\overset{\text{III}}{q} = \frac{a^3}{4}$, as in art. 7.

14.

From what is done in the laſt five articles, it evidently follows, that

$$-\overset{\text{IV}}{Q} \text{ is} = \frac{b^2\overset{\text{II}}{Q}}{2} + \frac{b^4}{2.2.3},$$

$$\frac{\overset{\text{V}}{q}}{\sqrt{-1}} = b\overset{\text{IV}}{Q} + \frac{b^3\overset{\text{II}}{Q}}{2.3} + \frac{b^5}{2.2.3.4},$$

$$-\overset{\text{VI}}{Q}\ \text{is} = \frac{b^2 \overset{\text{IV}}{Q}}{2} + \frac{b^4 \overset{\text{II}}{Q}}{2.3.4} + \frac{b^6}{2.\,.\,.3.4.5},$$

$$\frac{\overset{\text{VII}}{q}}{\sqrt{-1}} = b\overset{\text{VI}}{Q} + \frac{b^3 \overset{\text{IV}}{Q}}{2.3} + \frac{b^5 \overset{\text{II}}{Q}}{2.3.4.5} + \frac{b^7}{2.2.3.4.5.6},$$

&c. &c.

From whence (as well as from the theorems in art. 8.) may the values of $\overset{\text{IV}}{Q}$, $\overset{\text{V}}{q}$, $\overset{\text{VI}}{Q}$, $\overset{\text{VII}}{q}$, &c. be readily found, in terms of a.

15.

$\overset{\text{I}}{\dot{G}}$ being $= x + \frac{x^3}{3^2} + \frac{x^5}{5^2}$, &c. by art. 9.

$\overset{\text{I}}{H}$ = fluent of $x\dot{x}\overset{\text{I}}{G}$ is $= \frac{x^3}{1^2.3} + \frac{x^5}{3^2.5} + \frac{x^7}{5^2.7}$, &c.

$\overset{\text{II}}{H}$ = fluent of $\frac{\dot{x}}{x}\overset{\text{I}}{H} = \frac{x^3}{1^2.3^2} + \frac{x^5}{3^2.5^2} + \frac{x^7}{5^2.7^2}$, &c.

$\overset{\text{III}}{H}$ = fluent of $x\dot{x}\overset{\text{II}}{H} = \frac{x^5}{1^2.3^2.5} + \frac{x^7}{3^2.5^2.7} + \frac{x^9}{5^2.7^2.9}$, &c.

$\overset{\text{IV}}{H}$ = fluent of $\frac{\dot{x}}{x}\overset{\text{III}}{H} = \frac{x^5}{1^2.3^2.5^2} + \frac{x^7}{3^2.5^2.7^2} + \frac{x^9}{5^2.7^2.9^2}$, &c.

&c. &c. &c.

16.

Moreover, $\overset{\text{I}}{G}$ being $= 2\overset{\text{II}}{Q} + bX - x^{-1} - \frac{x^{-3}}{3^2} - \frac{x^{-5}}{5^2}$, &c. by art. 11. we, by multiplying by $x\dot{x}$, and taking the correct fluents, get $\overset{\text{I}}{H} = x^2\overset{\text{II}}{Q} - \overset{\text{II}}{Q} + \frac{bx^2X}{2} - \frac{bx^2}{4} + \frac{b}{4} - x + 1 + 2\overset{\text{II}}{S} + \frac{x^{-1}}{1.3^2} + \frac{x^{-3}}{3.5^2} + \frac{x^{-5}}{5.7^2}$, &c. $\overset{\text{II}}{S}$ being put for the ſeries $\frac{1}{1^2.3^2} + \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2}$, &c.

Now,

Now, it is obvious, that this value of $\overset{\mathrm{I}}{H}$ muſt be equal to the value of $\overset{\mathrm{I}}{H}$ in the preceding article, when both ſeries converge.

Therefore $\frac{3x^3 - x^{-1}}{1^2 \cdot 3^2} + \frac{5x^5 - 3x^{-3}}{3^2 \cdot 5^2} + \frac{7x^7 - 5x^{-5}}{5^2 \cdot 7^2}$, &c. is then $= x^2 \overset{\mathrm{II}}{Q} - \overset{\mathrm{II}}{Q} + \frac{bx^2 X}{2} - \frac{bx^2}{4} + \frac{b}{4} - x + 1 + 2\overset{\mathrm{II}}{S}$.

Hence, by taking x equal to -1, we find $-2\overset{\mathrm{II}}{S} = b^2 + 2 + 2\overset{\mathrm{II}}{S}$; and, conſequently, $\overset{\mathrm{II}}{S} = \frac{a^2}{4} - \frac{1}{2}$.

And, by taking x equal to $\frac{1}{\sqrt{-1}}$, we find

$$-\frac{2}{\sqrt{-1}} \times \overline{\frac{1}{1^2 \cdot 3^2} - \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} -, \&c.} = -2\overset{\mathrm{II}}{Q} - \frac{b^2}{2} + \frac{b}{2} - \frac{1}{\sqrt{-1}} + 1 + 2\overset{\mathrm{II}}{S} = \frac{a}{2\sqrt{-1}} - \frac{1}{\sqrt{-1}};$$

and, conſequently, $\frac{1}{1^2 \cdot 3^2} - \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2\,7^2} -$, &c. $= \frac{1}{2} - \frac{a}{4}$.

17.

Seeing that $\overset{\mathrm{II}}{Q}$ is $= \frac{a^2}{2}$, and $\overset{\mathrm{II}}{S} = \frac{a^2}{4} - \frac{1}{2}$, it follows, from the laſt article, that

$\overset{\mathrm{I}}{H}$ is $= x^2 \overset{\mathrm{II}}{Q} + \frac{bx^2 X}{2} - \frac{bX^2}{4} + \frac{b}{4} - x + \frac{x^{-1}}{1 \cdot 3^2} + \frac{x^{-3}}{3 \cdot 5^2} + \frac{x^{-5}}{5 \cdot 7^2}$, &c.

From whence, by multiplying by $\frac{\dot{x}}{x}$, and taking the correct fluent, we get

$$\overset{\text{II}}{H} = \frac{x^2 \overset{\text{II}}{Q}}{2} + \frac{b x^2 X}{4} - \frac{b x^2}{4} + \frac{b}{4} + \frac{b X}{4} - x + \frac{a^2}{4} - \frac{x^{-1}}{1^2 . 3^2} - \frac{x^{-3}}{3^2 . 5^2} - \frac{x^{-5}}{5^2 . 7^2}, \text{ \&c.}$$

And hence, by multiplying by $x \dot{x}$, and taking the correct fluents, we have

$$\overset{\text{III}}{H} = \frac{x^4 \overset{\text{II}}{Q}}{8} + \frac{b x^4 X}{16} - \frac{5 b x^4}{64} + \frac{b x^2}{16} + \frac{b x^2 X}{8} + \frac{b}{64} - \frac{x^3}{3} - \frac{x}{9} + \frac{4}{9} + \frac{a^2 x^2}{8} - \frac{3 a^2}{16} + 4 \overset{\text{III}}{S} + \frac{x^{-1}}{1 . 3^2 . 5^2} + \frac{x^{-3}}{3 . 5^2 . 7^2} + \frac{x^{-5}}{5 . 7^2 . 9^2}, \text{ \&c.}$$

$\overset{\text{III}}{S}$ being put for the ſeries $\frac{1}{1^2 . 3^2 . 5^2} + \frac{1}{3^2 . 5^2 . 7^2} + \frac{1}{5^2 . 7^2 . 9^2}$, &c.

Now, this value of $\overset{\text{III}}{H}$ being equal to the value of $\overset{\text{III}}{H}$ in art. 15. when both ſeries converge, it follows, that $\frac{5 x^5 - x^{-1}}{1^2 . 3^2 . 5^2} + \frac{7 x^7 - 3 x^{-3}}{3^2 . 5^2 . 7^2} + \frac{9 x^9 - 5 x^{-5}}{5^2 . 7^2 . 9^2}$, &c. is then

$$= \frac{x^4 \overset{\text{II}}{Q}}{8} + \frac{b x^4 X}{16} - \frac{5 b x^4}{64} + \frac{b x^2}{16} + \frac{b x^2 X}{8} + \frac{b}{64} - \frac{x^3}{3} - \frac{x}{9} + \frac{4}{9} + \frac{a^2 x^2}{8} + \frac{3 a^2}{16} - 4 \overset{\text{III}}{S}.$$

Hence, by taking x equal to -1, we find $-4 \overset{\text{III}}{S} = \frac{3 b^2}{8} + \frac{8}{9} + 4 \overset{\text{III}}{S}$; and, conſequently, $\overset{\text{III}}{S} = \frac{3 a^2}{64} - \frac{1}{9}$.

Many other inſtances of the uſe of this method might be given; but theſe may ſuffice to enable the intelligent reader to purſue the ſpeculation farther, at his pleaſure.